Derivatives and Maximization of Concave Functions

Let $f : S \to \mathbb{R}$ be a function on a convex set $S \subseteq \mathbb{R}^n$. The following theorems are provided without proofs. The parallel theorems for convex functions f follow immediately from these theorems by applying the theorems to the function $-f$.

Theorem: If f is concave, then a local maximum of f is a global maximum of f.

Theorem: If f is strictly concave, then it has at most one global maximum (*i.e.*, a global maximum is unique).

Definition: A subset of $\mathbb{R}^n$ is **open** if it contains a neighborhood of each of its points.

The most common examples of open sets are $\mathbb{R}^n$ itself and $\mathbb{R}^n_{++}$. An open ball about a point $\mathbf{x} \in \mathbb{R}^n$ is obviously an open set as well.

Theorem: If S is open and f is differentiable and concave on S, then $\overline{x}$ is a maximum of f on S if and only if $\nabla f(\overline{x}) = \mathbf{0}$.

Theorem: If S is open and f is twice continuously differentiable on S, then

(a) f is concave if and only if the Hessian matrix $D^2 f(\mathbf{x})$ is negative semidefinite at every $\mathbf{x} \in S$;

(b) if $D^2 f(\mathbf{x})$ is negative definite at every $\mathbf{x} \in S$, then f is strictly concave.